

Non-extremal multi-Kaluza-Klein black holes in five dimensions

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Abstract

Using a solution generating method, we derive an exact solution of the Einstein-Maxwell-Dilaton field equations in five dimensions describing charged multi-black hole configurations. More specifically, this solution describes systems of non-extremal charged static black holes with Kaluza-Klein asymptotics. As expected, we find that, in general, there are conical singularities in-between the Kaluza-Klein black holes that cannot be completely eliminated. We compute their conserved charges and investigate some of their properties.

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1 Introduction

Black holes in higher dimensions have been actively studied in recent years. Notably, with Emparan and Reall’s discovery of the five dimensional black ring solution [1], it was realized that higher dimensional black holes exhibit a much richer structure than their four-dimensional counterparts (for nice reviews of the black ring see [2] and of higher dimensional black holes see [3, 4]). In four-dimensional asymptotically flat space-times, as shown geometrically by Hawking [5], a black hole can have only spherical horizon topology; this result also follows under more general conditions as a consequence of topological censorship [6, 7]. However, in five dimensions, the spherical topology of infinity does not constrain that of the black hole horizon [8, 9]. Geometric considerations, however, restrict the horizon topology to those, such as S^3 and $S^2 \times S^1$, that admit positive scalar curvature [10]. In this regard, the black ring provides us with an explicit example of an asymptotically flat black hole having an $S^1 \times S^2$ horizon topology. Furthermore, it can carry (in certain conditions) the same amount of mass and angular momenta as the spherical Myers-Perry black hole [11]. Consequently, five-dimensional black holes are not uniquely characterised by their mass and angular momenta; the uniqueness theorems for black holes in four dimensions cannot be extended to the five dimensional case without further assumptions of additional symmetry and specification of the rod structure [12].

In five dimensions, there also exist the so-called squashed Kaluza-Klein (KK) black holes, whose horizon geometry is a squashed three-sphere [14, 15, 16, 17]. Their geometry is asymptotic to a non-trivial S^1 bundle over a four-dimensional asymptotically flat spacetime, which is also the asymptotic geometry of the Kaluza-Klein magnetic monopole [18, 19]. Such black holes look five-dimensional in the near-horizon region, while at infinity (asymptotically) they look like four-dimensional objects with a compactified fifth dimension. Again, uniqueness theorems for KK black holes are proven assuming additional symmetry and specification of other invariants [13]. Thus explicit examples of such solutions are valuable. KK black hole solutions in the presence of matter fields, are generally found by solving the Einstein equations by brute force. For instance, a solution describing a static KK black hole with electric charge has been found in [20], and the corresponding Einstein-Yang-Mills solution has been described in [21]. Remarkably, with hindsight, many such KK solutions can be generated from known solutions by applying a ‘squashing’ transformation on suitable geometries [22, 23, 24, 25, 26, 27]. However, not all of the KK black hole solutions can be generated by a squashing transformation; more general KK black holes have been derived in the context of the minimal 5-dimensional supergravity, [28, 29, 30].

In our work, we focus on multi-black hole solutions in spaces with Kaluza-Klein asymptotics. In general, in higher dimensions, solutions describing systems of charged multi-black holes are rare. Unlike the single black hole case, all known solution-generating techniques lead to solutions describing configurations formed either from extremal black holes [31, 32, 33, 34] or from non-extremal black holes with charges proportional to the masses. In five dimensions, a solution describing a general double-black hole configuration has been recently constructed in [35, 36], generalizing the uncharged solutions given in [37]. The solution generating technique from [35] has been further modified in [38] to obtain general multi-black hole solutions in spaces with KK asymptotics. A solution describing a system of two general KK black

holes in the double-Taub-NUT background has been explicitly constructed and studied in [38]. The purpose of the present work is to construct explicitly a general exact solution describing a superposition of N static Kaluza-Klein black holes with squashed horizons in five dimensions. We first consider the particular case of uncharged black holes. We describe some of its physical properties and, by using a standard charging technique, we obtain its generalization to a solution describing a configuration of static electrically charged squashed black holes, with fixed mass to charge ratios. Finally, we comment on how to obtain the most general solution of this kind in five dimensions.

2 The solution generating technique

Let us recall here the main results of the solution generating technique used in [38]. The main idea of this method is to map a general static electrically charged axisymmetric solution of Einstein-Maxwell theory in four dimensions to a five-dimensional static electrically charged axisymmetric solution of the Einstein-Maxwell-Dilaton (EMD) theory with arbitrary coupling of the dilaton to the electromagnetic field. To this end one performs first a dimensional reduction of both theories down to three dimensions and, after a careful comparison of the dimensionally-reduced lagrangians and mapping of the scalar fields and electromagnetic potentials, one is able to bypass the actual solving of the field equations by algebraically mapping solutions of one theory to the other. More precisely, suppose that we are given a static electrically charged solution of the four-dimensional Einstein-Maxwell system with Lagrangian

$$\mathcal{L}_4 = \sqrt{-g} \left[R - \frac{1}{4} \tilde{F}_{(2)}^2 \right], \quad (1)$$

where $\tilde{F}_{(2)} = d\tilde{A}_{(1)}$ and the only non-zero component of $\tilde{A}_{(1)}$ is $\tilde{A}_t = \Psi$. The solution to the equations of motion derived from (1) is assumed to have the following static and axisymmetric form:

$$ds_4^2 = -\tilde{f}dt^2 + \tilde{f}^{-1} [e^{2\tilde{\mu}}(d\rho^2 + dz^2) + \rho^2 d\varphi^2], \quad \tilde{A}_{(1)} = \Psi dt. \quad (2)$$

Here and in what follows we assume that all the functions \tilde{f} , $\tilde{\mu}$, and Ψ depend only on coordinates ρ and z .

Then the corresponding solution of the Einstein-Maxwell-Dilaton system in five dimensions with Lagrangian

$$\mathcal{L}_5 = \sqrt{-g} \left[R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{4} e^{\alpha\phi} F_{(2)}^2 \right] \quad (3)$$

where $F_{(2)} = dA_{(1)}$ can be written as:

$$ds_5^2 = -\tilde{f}^{\frac{4}{3\alpha^2+4}} dt^2 + \tilde{f}^{-\frac{2}{3\alpha^2+4}} \left[\frac{e^{2h}}{a^2 - c^2 e^{4h}} (d\chi + 4acHd\varphi)^2 + (a^2 - c^2 e^{4h}) e^{\frac{6\tilde{\mu}}{3\alpha^2+4} + 2\gamma - 2h} (d\rho^2 + dz^2) + \rho^2 (a^2 - c^2 e^{4h}) e^{-2h} d\varphi^2 \right], \quad A_{(1)} = \sqrt{\frac{3}{3\alpha^2+4}} \Psi dt, \quad e^{-\phi} = \tilde{f}^{\frac{3\alpha}{3\alpha^2+4}}. \quad (4)$$

Here a and c are constants, while h is an arbitrary harmonic function¹. Once the form of h has been specified for a particular solution, the remaining function γ can be obtained by simple quadratures using the equations:

$$\partial_\rho \gamma = \rho[(\partial_\rho h)^2 - (\partial_z h)^2], \quad \partial_z \gamma = 2\rho(\partial_\rho h)(\partial_z h). \quad (5)$$

Also, the function H is the so-called ‘dual’ of h and it is a solution of the following equation:

$$dH = \rho(\partial_\rho h dz - \partial_z h d\rho). \quad (6)$$

Solutions of the pure Einstein-Maxwell theory in five dimensions are simply obtained from the above formulae by taking $\alpha = 0$. In the following sections we shall focus on this particular case.

3 The vacuum multi-KK black hole solution

It has been shown in [38] that if one starts from the four-dimensional Reissner-Nordström black hole and use the above solution generating technique one is able to recover the five-dimensional charged KK black hole after setting $a^2 = 1 + c^2$. In this case the harmonic function h is a ‘correction’ function that depends on the presence of a black hole horizon in the initial seed solution. Then one expects that, in order to construct the five-dimensional generalization of the multi KK black hole solution, one should make use of the solution previously constructed by Israel and Khan [39] which describes multiple collinear Schwarzschild black holes connected by rods. It turns out that this is indeed the case. In our solution generating technique, the form of the harmonic function h will now correspond to correction factors applied for each black hole horizon in the four-dimensional Israel-Khan solution. In terms of the ansatz given in (2), the Israel-Khan solution that describes N collinear Schwarzschild black holes is given by:

$$\tilde{f} = \prod_{i=1}^N \frac{r_{2i-1} + \zeta_{2i-1}}{r_{2i} + \zeta_{2i}}, \quad e^{2\tilde{\mu}} = \frac{1}{K_0} \left(\frac{4^N}{r_1 \dots r_N} \frac{\prod_{i,j=1}^N Y_{2i-1,2j}}{\prod_{i=1}^N \prod_{k>i} Y_{2i,2k} \prod_{i=1}^N \prod_{k>i} Y_{2i-1,2k-1}} \right). \quad (7)$$

Here we generally denote $\zeta_i = z - a_i$, $r_i = \sqrt{\rho^2 + \zeta_i^2}$ while $Y_{i,j} = r_i r_j + \zeta_i \zeta_j + \rho^2$, while K_0 is an arbitrary constant, fixed in four dimensions by requiring that the asymptotic geometry be flat.² This solution describes then a system of N collinear black holes, having the rods corresponding to the black hole horizons depicted in Figure 1.

In five dimensions, to describe a configuration of N KK black holes one has to pick the following harmonic function:

$$e^{2h} = \prod_{i=1}^N \sqrt{\frac{r_{2i-1} + \zeta_{2i-1}}{r_{2i} + \zeta_{2i}}}. \quad (8)$$

¹That is, it satisfies the equation $\nabla^2 h = \frac{\partial^2 h}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial h}{\partial \rho} + \frac{\partial^2 h}{\partial z^2} = 0$.

²In what follows we shall keep it unconstrained in the seed solution.

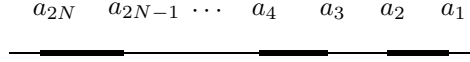


Figure 1: Rod structure of the multi-black hole system.

Noting that $e^{2h} = \tilde{f}^{\frac{1}{2}}$, one can actually bypass the integration of (5) by using the scaling symmetry from [40] to obtain the particularly simple result $\tilde{\mu} = 4\gamma$.

Finally, the dual of h turns out to have the particularly simple form³:

$$H = \frac{1}{4} \sum_{i=1}^N (r_{2i} - r_{2i-1}). \quad (9)$$

To summarize, denoting by $\Sigma = 1 + c^2(1 - \tilde{f})$, the final five-dimensional solution describing a system of N uncharged KK black holes is given by:

$$ds^2 = -\tilde{f}dt^2 + \frac{1}{\Sigma} \left(d\chi + ac \sum_{i=1}^N (r_{2i} - r_{2i-1}) d\varphi \right)^2 + \frac{\Sigma}{\tilde{f}} [e^{2\tilde{\mu}}(d\rho^2 + dz^2) + \rho^2 d\varphi^2]. \quad (10)$$

One can easily check that for $N = 2$ this solution corresponds to the uncharged version of the double KK black hole solution previously constructed in [38].

Let us consider now the rod structure of this general solution. For simplicity, let us denote the rod length of each black hole horizon by $2\sigma_i = a_{2i-1} - a_{2i}$. Following the procedure given in [41, 42] one deduces that the rod structure of the general solution is described by $2N$ turning points that divide the z -axis into $2N + 1$ rods as follows.⁴ For $z < a_{2N}$ such that all $\zeta_i < 0$ one has a semi-infinite rod with direction $l_1 = (0, 2ac \sum_{i=1}^N \sigma_i, 1)$. For $a_{2N} < z < a_{2N-1}$ one has a finite timelike rod with direction $l_2 = (1, 0, 0)$, corresponding to the horizon of the N -th black hole. For $a_{2N-1} < z < a_{2N-2}$ one has a spacelike rod with direction $l_3 = (0, 2ac(\sigma_N - \sum_{i=1}^{N-1} \sigma_i), 1)$. More generally, for each black hole horizon $a_{2i} < z < a_{2i-1}$ one has a timelike rod $(1, 0, 0)$, while in between the black holes (for instance for $a_{2j-1} < z < a_{2j-2}$, which is the rod in between the $(j-1)$ -th black hole and the j -th black hole) one has a finite spacelike rod $l_{2(N-j)+3} = (0, 2ac(-\sum_{i=1}^{j-1} \sigma_i + \sum_{i=j}^N \sigma_i), 1)$. Finally, for $z > a_1$ one has a semi-infinite spacelike rod with direction $l_{2N+1} = (0, -2ac \sum_{i=1}^N \sigma_i, 1)$.

Note now that the rod directions of the spacelike rods surrounding the horizons are precisely the rod directions of the multi-Taub-NUT background [43]. This confirms that the general solution that we derived describes a configuration of black holes sitting at the nuts of the multi-collinearly-centered Taub-NUT background. One can also recover directly the multi-Taub-NUT background by taking the limit in which the black hole horizons disappear. For this, recall that $a^2 = 1 + c^2$ and let us take the limit $c \rightarrow \infty$ and $\sigma_i \rightarrow 0$ such that $N_i = c^2 \sigma_i$ are kept constant. Then, if one denotes $a_{2i-1} = b_i + \sigma_i$ and $a_{2i} = b_i - \sigma_i$ (such that the i -th black hole horizon is centered at b_i) by expanding to first order in σ_1 and σ_2

³Up to a constant. In general, the dual of $\frac{1}{2} \ln(r_i + \zeta_i)$ is given by $-\frac{1}{2}(r_i - \zeta_i)$.

⁴We are writing the vectors in the basis $\{\partial/\partial t, \partial/\partial \chi, \partial/\partial \varphi\}$.

one obtains:

$$\Sigma = 1 + \sum_{i=1}^N \frac{N_i}{\sqrt{\rho^2 + (z - b_i)^2}} + \mathcal{O}(\sigma_i^2), \quad (11)$$

while

$$acH = \sum_{i=1}^N \frac{N_i(z - b_i)}{\sqrt{\rho^2 + (z - b_i)^2}} + \mathcal{O}(\sigma_i^2). \quad (12)$$

Since in absence of the black holes $\tilde{f} = 1$, it is now clear that one recovers as background the multi-collinearly-centered Taub-NUT space with a trivial time direction, as advertised.

We now turn to the discussion of the conical singularities. To avoid a conical singularity at the location of a rod with direction l_i , the period Δ_i of the spacelike coordinate η_i (such that $l_i = \partial/\partial\eta_i$) must be fixed as:

$$\Delta_i = 2\pi \lim_{\rho \rightarrow 0} \sqrt{\frac{\rho^2 g_{\rho\rho}}{|l_i|^2}}, \quad (13)$$

where $g_{\rho\rho}$ is the $\rho\rho$ -component of the metric while $|l_i|^2$ is the norm of l_i . More specifically, for the outer axis one has:

$$\Delta_1 = \Delta_{2N+1} = 2\pi \sqrt{\frac{2^{N^2}}{K_0}}, \quad (14)$$

and the conical singularities can be eliminated there by picking $K_0 = 2^{N^2}$. However, in-between the black holes, the expressions for $\Delta_i = 2\pi e^{\tilde{\mu}}|_{\rho \rightarrow 0}$ are much more complicated and not informative to list here. It turns out that the conical singularities in-between the black holes cannot be eliminated for any physically reasonable values of the parameters describing the solution. This is, in fact, expected since the multi-black hole solution is static and there are no other forces that could counteract the gravitational attraction between the black holes.

4 Multiple charged KK black holes

The solution generating technique described in Section 2 allows us to construct directly the solution describing the superposition of N charged KK black holes; one starts instead with the superposition of N charged Reissner-Nordström black holes in four dimensions [44]. In this case, the harmonic function h will be again given by (8) while the expression for $\gamma = \tilde{\mu}/4$ can be easily read from (7). However, given the very complicated form of the general solution describing N charged black holes we have chosen to consider here the particular case in which the mass-to-charge ratio is fixed for each black hole. Such a solution can be easily obtained from the uncharged version presented in the previous section by applying a charging technique. One particularly simple charging technique has been described in [45, 46].

Considering again (for simplicity) the case in which the dilaton field is turned off in (3), starting from the vacuum solution describing a configuration of N KK black holes one can obtain its charged version in the following form:

$$ds^2 = -\Omega^{-2}\tilde{f}dt^2 + \Omega\left[\frac{1}{\Sigma}\left(d\chi + ac\sum_{i=1}^N(r_{2i} - r_{2i-1})d\varphi\right)^2 + \frac{\Sigma}{\tilde{f}}[e^{2\tilde{\mu}}(d\rho^2 + dz^2) + \rho^2d\varphi^2]\right],$$

$$A_t = \sqrt{3}\frac{\tilde{f}U}{\Omega}, \quad \text{where} \quad \Omega = \frac{1 - U^2\tilde{f}}{1 - U^2}, \quad (15)$$

while U is the parameter of the Harrison transformation, with $0 \leq U < 1$. For $U = 0$ one recovers the vacuum configuration, while $U \rightarrow 1$ corresponds to taking the extremal limit of this charged solution.

Following the analysis performed for the double-black hole case, one is able to compute some of the conserved charges for this multi-black hole configuration. The total mass and the total electric charge are computed in the asymptotic region, which is reached by first performing the following coordinate transformations:

$$\rho = r \sin \theta, \quad z = r \cos \theta, \quad (16)$$

and taking the $r \rightarrow \infty$ limit. Defining now $r_\infty = c\sqrt{1+c^2}\sum_{i=1}^N(2\sigma_i)$ the asymptotic length of the χ -circle becomes $\mathcal{L} = 4\pi r_\infty$. Using the counterterm approach [47, 46] one is now ready to compute the total mass, the gravitational tension and the total electric charge for this configuration:

$$\mathcal{M} = \frac{\mathcal{L}}{4G} \frac{2 + U^2 + c^2(1 - U^2)}{1 - U^2} \sum_{i=1}^N 2\sigma_i, \quad \mathcal{T} = \frac{1 + 2c^2}{4G} \sum_{i=1}^N 2\sigma_i, \quad \mathcal{Q} = \frac{\sqrt{3}\mathcal{L}}{4G} \frac{U}{1 - U^2} \sum_{i=1}^N 2\sigma_i.$$

One can also compute the so-called Komar mass, check that $2M_K = 2\mathcal{M} - \mathcal{T}\mathcal{L}$ and verify that the Smarr relation is satisfied:

$$2M_K = 3 \sum_{i=1}^N \frac{A_{(5)}^i k_{(5)}^i}{8\pi G} + 2\Phi \mathcal{Q}, \quad (17)$$

where $\Phi = \Phi_i = \sqrt{3}U$ is the electric potential of each black hole horizon, while for each black hole one also has:

$$\frac{A_{(5)}^i k_{(5)}^i}{8\pi G} = \frac{2\sigma_i \mathcal{L}}{4G}, \quad (18)$$

where $A_{(5)}^i$ is the horizon area of the i -th black hole, while $k_{(5)}^i$ is its surface gravity.

As we have mentioned, the extremal limit is obtained in the limit $U \rightarrow 1$ (such that the value of the electric charge is kept finite), which amounts to keeping $M_i = \frac{2\sigma_i}{1-U^2}$ fixed. On the other hand, once $\sigma_i \rightarrow 0$ one also has to take the limit $c \rightarrow \infty$ such that $N_i = c^2\sigma_i$ are kept fixed in order to preserve the black holes on the multi-collinearly Taub-NUT background. Gathering up all these results, the extremal solution reduces to:

$$ds^2 = -\Omega_e^{-2}dt^2 + \Omega_e \left[\Sigma_e^{-1}(d\chi + \omega d\varphi)^2 + \Sigma_e (d\rho^2 + dz^2 + \rho^2 d\varphi^2) \right], \quad A_t = \sqrt{3}\Omega_e^{-1}, \quad (19)$$

where:

$$\Omega_e = 1 + \sum_{i=1}^N \frac{M_i}{\sqrt{\rho^2 + (z - b_i)^2}}, \quad \Sigma_e = 1 + \sum_{i=1}^N \frac{N_i}{\sqrt{\rho^2 + (z - b_i)^2}}, \quad \omega = \sum_{i=1}^N \frac{N_i(z - b_i)}{\sqrt{\rho^2 + (z - b_i)^2}}.$$

This is indeed the extremal multi-KK black hole solution derived previously in [33].

5 Conclusions

The main purpose of this work was to explicitly derive an exact solution describing a superposition of N charged KK black holes in five dimensions. While the general solution of the EMD system is given in (4), when discussing the generated solutions, we focused for simplicity on the Einstein-Maxwell theory, for which the coupling constant $\alpha = 0$ in the general solution (4) vanishes.

One should note that the solution generating technique that we used allows us to easily construct the most general solution describing a collinear superposition of N charged KK black holes in five dimensions. For this purpose one should use as the seed solution the general metric constructed previously in [44] in four dimensions. In this case the harmonic function h is the same as the one used in the vacuum case in Section 3. However, due to the complexity of the four-dimensional seed solution and mostly for simplicity reasons, we have chosen to discuss here two particular cases. In the third section of this article we focused on the particular case of N neutral KK black holes and studied some of its properties. In particular, we showed that the rod directions of the spacelike rods surrounding the black hole horizons correspond precisely to those of the multi-collinearly-centered Taub-NUT background. We also showed explicitly that in the absence of black holes, one recovers the multi-Taub-NUT background. Finally, in Section 4 we discussed the particular case of N charged KK black holes having fixed mass-to-charge ratio. To generate such a solution from the uncharged version we made use of a charging technique previously discussed in [45, 46]. By using a counterterm approach we computed the total mass, electric charge and gravitational tension and we showed that the Smarr relation for this configuration is satisfied, as expected. Finally, we showed how to recover in the extremal limit the multi-KK black hole solution previously derived in [33].

As avenues for further work, it would be interesting to extend the thermodynamic analysis of [48] to five dimensions and find an embedding of the KK multi-black hole solution in string theory. Using the effective string description, one should be able to compute the entropy, including the corrections associated with the interaction among the black holes. Since the static multi-black hole solution necessarily contains conical singularities that cannot be eliminated, to properly study the thermodynamic properties of such a configuration one should make use of the recent formalism developed in [49, 50] that was specifically devised to deal with such systems. However, we leave these issues for further work.

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